The Matter Bounce Scenario in Loop Quantum Cosmology

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In the matter bounce scenario, a dust-dominated contracting space-time generates scale-invariant perturbations that, assuming a nonsingular bouncing cosmology, propagate to the expanding branch and set appropriate initial conditions for the radiation-dominated era. Since this scenario depends on the presence of a bounce, it seems appropriate to consider it in the context of loop quantum cosmology where a bouncing universe naturally arises. It turns out that quantum gravity effects play an important role beyond simply providing the bounce. Indeed, quantum gravity corrections to the Mukhanov-Sasaki equations significantly modify some of the results obtained in a purely classical setting: while the predicted spectra of scalar and tensor perturbations are both almost scale-invariant with identical small red tilts in agreement with previous results, the tensor to scalar ratio is now expected to be $r \approx 9 \times 10^{-4}$, which is much smaller than the original classical prediction. Finally, for the predicted amplitude of the scalar perturbations to agree with observations, the critical density in loop quantum cosmology must be of the order $\rho_c \sim 10^{-9} \rho_{\rm Pl}$.

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I. INTRODUCTION

Observations of the cosmic microwave background (CMB) have shown that the spectrum of scalar perturbations is scale-invariant with a slight red tilt [1] and therefore this is one of the major predictions that any viable cosmological model must make.

There are two well known mechanisms that generate scale invariant perturbations starting from the fluctuations of an initial quantum vacuum state: inflation, an exponential expansion of the universe, and a dust-dominated contracting phase. Recent reviews of these two paradigms are [2] and [3], respectively. The fact that they both give rise to a scale-invariant spectra of scalar and tensor perturbations is easily understood as they are related by a simple duality [4].

In this paper, we will focus on the second mechanism. More specifically, we will study the matter bounce scenario where an initially classical contracting dust-dominated universe with quantum vacuum fluctuations gives scale-invariant perturbations once the relevant modes exit the Hubble radius. Assuming the presence of a bounce, the scale-invariant perturbations then provide appropriate initial conditions for the expanding radiation-dominated epoch of our universe.

However, the standard classical treatment of the matter bounce has a major shortcoming in that one must evolve the perturbations from the pre-bounce era to the post-bounce one by hand. Typically this is done by using some reasonable matching conditions [5], but it would be nice to go beyond this and explicitly calculate the propagation of the perturbations through the bounce, where quantum gravity effects are expected to be important. Now, since the matter bounce scenario depends on the presence of a bounce, it is natural to work in the context of loop quantum cosmology (LQC) which predicts a bouncing universe. In this paper, we will study the matter bounce in LQC in order to determine the consequences of quantum gravity effects in this setting. As we shall see, some of the results obtained here are significantly different from those obtained in the matching procedure of [5]; these differences are due to modifications of the Friedmann and Mukhanov-Sasaki equations from quantum gravity effects.

Homogeneous LQC is obtained by following loop quantum gravity (LQG) and using holonomy and flux operators in order to quantize the Hamiltonian constraint corresponding to a homogeneous and isotropic space-time. See [6, 7] for recent reviews of LQC. One of the main results of LQC is that the classical big-bang singularity is resolved and replaced by a bounce that occurs when the space-time curvature is approximately of the Planck scale. In addition, careful studies of the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies in LQC have shown that, for states that are semi-classical (i.e., sharply peaked around a classical solution at late times), there exist a set of effective equations that provide an excellent approximation to the full quantum dynamics of the states at all times, even during the bounce [8, 9].

Cosmological perturbations have also been studied in some detail in LQC, both at the quantum level [10] and especially in the effective theory [11–14]. In this paper we will work with the effective equations for the sake of simplicity. This clearly requires the assumption that the effective equations continue to be a good approximation to the quantum dynamics of semi-classical states in the presence of linear perturbations, which seems reasonable so long as the perturbations remain small.

As an aside, we point out that there are two types of corrections, holonomy and inverse triad, that are considered in effective studies in LQC. In this paper we focus

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on holonomy corrections as they are the dominant ones in the quantum theory of homogeneous LQC and we assume this will continue to be the case when linear perturbations are included. See however [15] for a different perspective.

The paper is organized as follows: in Sec. II we review the effective theory in LQC for both the homogeneous FLRW background and the perturbations. Then we study the dynamics of the cosmological perturbations in a bouncing dust-dominated FLRW universe; scalar perturbations in Sec. III and tensor perturbations in Sec. IV. We close with a discussion in Sec. V.

II. LOOP QUANTUM COSMOLOGY

We start by briefly introducing the ingredients coming from loop quantum cosmology that will be necessary for the calculations in the following sections. Since we are only interested in space-times that are classical at times well before and well after the bounce, we can restrict our attention to semi-classical states in LQC. Because of this we can work in the effective theory; this will considerably simplify the analysis.

We will review the effective equations for the flat FLRW background in the first part of this section, and the effective theory for the scalar and tensor perturbations in the second part.

A. The Homogeneous Background

Given the flat FLRW metric in terms of the proper time,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\vec{x}^2, \tag{1}$$

where $a(t)$ is the scale factor, and denoting the energy density of the matter field by ρ and its pressure by P, the LQC effective equations for the background, including corrections due to quantum geometry effects, are

$$H^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_c}\right), \tag{2}$$

$$\frac{\ddot{a}}{a} - H^2 = -4\pi G\left(\rho + P\right)\left(1-\frac{2\rho}{\rho_c}\right), \tag{3}$$

$$\dot{\rho} = -3H\left(\rho + P\right), \tag{4}$$

where ρ_c is the critical energy density and $H = \dot{a}/a$ is the Hubble rate. The dot denotes differentiation with respect to the proper time t. Finally, note that the classical Friedmann equations are recovered in the limit of $\rho_c \to \infty$.

When the matter field is a perfect fluid with a constant equation of state $P = \omega\rho$, these equations can be solved, giving

$$\rho = \rho_o a^{-3(1+\omega)}, \tag{5}$$

$$a(t) = \left(6\pi G\rho_o(1+\omega)^2(t-t_o)^2 + \frac{\rho_o}{\rho_c}\right)^{\frac{1}{3(1+\omega)}}. \tag{6}$$

From the nonvanishing form of $a(t)$, it is clear that the big bang singularity is resolved and replaced by a bounce. The dimensionful constants of integration ρ_o and t_o, which determine the magnitude of the scale factor and the time the bounce occurs at, will be set to $\rho_o = \rho_c$ and $t_o = 0$ so that $a(t_{\text{bounce}}) = 1$ and $t_{\text{bounce}} = 0$; these choices do not affect the physics.

As an aside, it is worth pointing out that while the scale factor a can be expressed in a relatively simple manner in terms of the proper time t, this is not true when one works in conformal time η, defined by

$$\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}, \tag{7}$$

in which case the form of the scale factor is considerably more complicated. For this reason, we will try to work in proper time whenever LQC effects are important. On the other hand, it will be useful to work in conformal time in order to study the perturbations in the classical limit and therefore we will switch between the two different time choices depending on the situation.

Finally, if one is working with a scalar field, then the energy density and the pressure are given by the same relations as in the classical theory,

$$\rho = \frac{1}{2}\,\dot{\varphi}^2 + V(\varphi), \qquad P = \frac{1}{2}\,\dot{\varphi}^2 - V(\varphi). \tag{8}$$

It is possible, for the potential[1]

$$V(\varphi) = \frac{(1-\omega)V_o e^{-\sqrt{24\pi G(1+\omega)}\varphi}}{1+\frac{V_o}{2\rho_c}e^{-\sqrt{24\pi G(1+\omega)}\varphi}}, \tag{9}$$

to mimic a cosmology with a constant equation of state $P = \omega\rho$ by using a scalar field, so long as one starts with appropriate initial conditions. This is what will be done here, for the case of $\omega = 0$.

B. Perturbations

Classically, scalar perturbations in cosmology can be studied by using the Mukhanov-Sasaki equation. (For an introduction to the theory of cosmological perturbations in the classical theory, see for example [16, 17].) In

[1] Note that the denominator in the potential is necessary due to the modifications in the LQC Friedmann equations. The usual potential used to get a constant equation of state given in e.g. [5] is obtained in the limit $\rho_c \to \infty$.

LQC, there are modifications to the equations of motion governing the dynamics of the perturbations, just as the Friedmann equations are modified. The corrections have been studied for the cases of a perfect fluid [11] and a scalar field [12, 13]. Here we are working with a scalar field, in which case the modified Mukhanov-Sasaki equation is

$$v'' - \left(1 - \frac{2\rho}{\rho_c}\right) \nabla^2 v - \frac{z''}{z} v = 0, \tag{10}$$

where the prime denotes differentiation with respect to the conformal time η,

$$v = a\sqrt{\rho + P}\,\delta u^{(gi)} - z\Phi, \tag{11}$$

where $\delta u^{(gi)}$ and Φ are the usual gauge-invariant observables in cosmological perturbation theory and

$$z = \frac{a}{H}\sqrt{(\rho + P)}. \tag{12}$$

The Mukhanov-Sasaki equation is most easily solved in Fourier space, where it becomes

$$v_k'' + \left(1 - \frac{2\rho}{\rho_c}\right) k^2 v_k - \frac{z''}{z} v_k = 0. \tag{13}$$

Tensor perturbations behave a little differently: their dynamics are governed by an equation that has the same form as for scalar perturbations, but even classically the variables are defined slightly differently, and it turns out that the quantum geometry corrections do not appear in exactly the same manner either. The holonomy-corrected Mukhanov-Sasaki equation for tensor perturbations in Fourier space is given by [14]

$$\mu_k'' + \left(1 - \frac{2\rho}{\rho_c}\right) k^2 \mu_k - \frac{z_T''}{z_T} \mu_k = 0, \tag{14}$$

where

$$z_T = \frac{a}{\sqrt{1 - 2\rho/\rho_c}}, \tag{15}$$

and $\mu_k = z_T h_k$, with h being the usual gravitational wave perturbation variable. Note that there is no need for absolute values in the denominator of z_T as μ_k is a complex variable. The fact that there are no absolute values will be important in Sec. IV.

For ease of notation, we will drop the index k in (13) and (14) in the future.

III. SCALAR PERTURBATIONS

We will begin by studying the propagation of scalar perturbations in an LQC dust-dominated universe. As perturbations are most easily studied in conformal time, and the initial conditions are to be set in a classical regime where quantum gravity effects are negligible, it is useful to recall the classical relations giving the scale factor and the proper time in terms of the conformal time for $P = 0$,

$$a(\eta) = \frac{2\pi G \rho_c}{3}\,\eta^2, \qquad t(\eta) = \frac{2\pi G \rho_c}{9}\,\eta^3. \tag{16}$$

Another useful relation is $z(\eta)$ in the classical limit, given by

$$z(\eta) = \sqrt{\frac{\pi G}{6}}\,\rho_c\,\eta^2. \tag{17}$$

We will now study how the perturbations propagate through the bounce by imposing quantum vacuum initial conditions in the pre-bounce phase and then determining the form of the perturbations in the post-bounce phase. This will be done by matching the classical solutions, in both the pre- and post-bounce eras, order by order with a formal solution given by an expansion in k that holds at all times, including at the bounce point.

A. The Contracting Branch and the Bounce

In the classical limit, and for a dust-dominated universe, (13) simplifies to

$$v'' + \left(k^2 - \frac{2}{\eta^2}\right) v = 0. \tag{18}$$

If (18) is rewritten for $f = v/\sqrt{-\eta}$ and the time variable is rescaled by a factor of k, this becomes the Bessel differential equation for f, and thus the solution to (18) is given by

$$v(\eta) = \sqrt{-\eta}\left[A_1 H^{(1)}_{3/2}(-k\eta) + A_2 H^{(2)}_{3/2}(-k\eta)\right], \tag{19}$$

where $H^{(1)}_n(x)$ and $H^{(2)}_n(x)$ are the Hankel functions, while A_1 and A_2 are constants to be determined by the initial conditions. Note the presence of a minus sign in front of η as this solution holds in the contracting branch, where the time variables are negative.

Now, by the asymptotic behaviour of the Hankel functions, we find that (up to an irrelevant global phase which we leave out)

$$\lim_{\eta \to -\infty} v(\eta) \sim A_1 \sqrt{\frac{2}{\pi k}}\, e^{-ik\eta} + A_2 \sqrt{\frac{2}{\pi k}}\, e^{ik\eta}, \tag{20}$$

and therefore it is possible to impose the vacuum initial conditions

$$v_{\text{initial}} = \sqrt{\frac{\hbar}{2k}} e^{-ik\eta}, \tag{21}$$

by choosing $A_1 = \sqrt{\pi\hbar/4}$ and $A_2 = 0$, which gives

$$v(\eta) = \sqrt{\frac{-\pi\hbar\eta}{4}} H^{(1)}_{3/2}(-k\eta). \tag{22}$$

This gives the expression of the Mukhanov variable so long as quantum geometry effects are negligible.

Of course, as the space-time contracts, the curvature will increase and at some point LQC effects will begin to play an important role. Therefore it will be necessary to match this solution to another one which will be valid throughout the bounce.

This can be done by rewriting (13) in an integral form, i.e., as

$$\begin{aligned} v(\eta) =& B_1 z + B_2 z \int^\eta \frac{\mathrm{d}\bar\eta}{z^2} - k^2 z \int^\eta \frac{\mathrm{d}\bar\eta}{z^2} \int^{\bar\eta} \mathrm{d}\bar{\bar\eta}\, z\, v \\ &+ \frac{2k^2}{\rho_c} z \int^\eta \frac{\mathrm{d}\bar\eta}{z^2} \int^{\bar\eta} \mathrm{d}\bar{\bar\eta}\, \rho\, z\, v. \end{aligned} \tag{23}$$

Doing an expansion in k, we immediately find that the leading order terms are

$$v(\eta) = B_1 \left(z + O(k^2)\right) + B_2 \left(z \int^\eta \frac{\mathrm{d}\bar\eta}{z^2} + O(k^2) \right). \tag{24}$$

An important point here is that the constants B_1 and B_2 can (and will) depend on k. For a dust-dominated space-time in LQC,

$$a(t) = \left(6\pi G \rho_c t^2 + 1\right)^{1/3}, \quad z(t) = \frac{a(t)^{5/2}}{4\pi G \sqrt{\rho_c}\, t}, \tag{25}$$

and by using (7) we find that the two leading order terms in k of $v(t)$ are given by

$$\begin{aligned} v(t) =& B_1 z(t) + \left[\sqrt{\frac{8\pi G}{27\rho_c}} \left(\arctan \sqrt{6\pi G\rho_c} t + \frac{\pi}{2} \right) \right. \\ & \left. - \frac{4\pi G t}{3\left(6\pi G \rho_c t^2 + 1\right)} \right] B_2 z(t), \end{aligned} \tag{26}$$

where the constant of integration has been chosen in order to simplify the matching of this solution with (22). Indeed, in the classical regime of the contracting branch where $t \ll -1/\sqrt{6\pi G \rho_c}$, we find that

$$v(t) = B_1 z(t) - \frac{4 B_2 z(t)}{9 \rho_c t}, \tag{27}$$

and it is possible to match this solution with (22) by ensuring that the coefficients of the leading terms in k match in the classical regime.

We cannot take the limit $\eta \to 0$ of (22) as quantum gravity effects are important in this regime and then the classical solution cannot be trusted. Instead, we will consider the limit of small $k\eta$ at a time where general relativity can be trusted. Note that this limit can only be taken for the modes that have become larger than the Hubble radius before LQC effects become important. However, in the matter bounce scenario all of the modes that we can observe in the CMB today satisfy this property and therefore this limitation is not problematic. In the small $k\eta$ limit, $v(\eta)$ contributes two terms, each of which can be used to fix one of B_1 and B_2.

In the $k\eta \ll 1$ limit, the solution (22) tends to

$$v(\eta) \to \frac{\sqrt{\hbar}}{3\sqrt{2}}\, k^{3/2} \eta^2 - \frac{i\sqrt{\hbar}}{\sqrt{2}\, k^{3/2}\, \eta}, \tag{28}$$

and therefore the constants B_1 and B_2 must be taken to be

$$B_1 = \frac{\sqrt{\hbar}}{\sqrt{3\pi G \rho_c}}\, k^{3/2}, \quad B_2 = i \frac{\sqrt{3\pi G \hbar}}{2}\, \rho_c\, k^{-3/2}. \tag{29}$$

B. The Expanding Branch

We now know the form of $v(t)$, at least in the form of the formal solution (23), and from this it is possible to determine the spectrum of scalar perturbations after the bounce.

In order to do this, we will again work with the leading order terms of the formal solution to v, given by (26). For times after the bounce where quantum gravity effects are negligible, $t \gg 1/\sqrt{6\pi G \rho_c}$ and then the behaviour of $v(t)$ is, for the leading order terms in k,

$$v(t) = \left(B_1 + \sqrt{\frac{8\pi^3 G}{27 \rho_c}} B_2 \right) z(t) - \frac{4 B_2 z(t)}{9 \rho_c t}, \tag{30}$$

and we see that a mixing has occurred in the first term. It is this new term, coming from the mixing, that will give a scale-invariant spectrum. This is what we shall show now.

In order to have the full solution to $v(\eta)$, the easiest method is to use the classical Mukhanov-Sasaki equation (18) that holds once LQC effects become small after the bounce. The classical solution, to all orders in k, is given by

$$v(\eta) = \sqrt{\eta} \left[C_1 H^{(1)}_{3/2}(k\eta) + C_2 H^{(2)}_{3/2}(k\eta) \right], \tag{31}$$

where C_1 and C_2 are two constants that must be determined by matching the lowest order terms in k of this solution with (30).

This matching gives

$$v(\eta) = \sqrt{\pi \hbar \eta} \left[\frac{1}{2} H^{(2)}_{3/2}(k\eta) + i \frac{\pi^2 (G\rho_c)^{3/2}}{\sqrt{6}\, k^3} J_{3/2}(k\eta) \right], \tag{32}$$

where $J_n(x)$ is the Bessel function of the first kind and it is understood that the prefactors have been determined to leading order in k.

Now we will assume that there is a transition to a radiation-dominated epoch after quantum gravity effects are negligible, but before the modes reenter the Hubble

radius. Then, the perturbations just before this transition point look like

$$v(\eta) = \frac{i\sqrt{\hbar}}{k^{3/2}} \left[\pi^{5/2} \left(\frac{G\rho_c}{3} \right)^{3/2} \eta^2 - \frac{1}{\sqrt{2}\pi\eta} \right] + \frac{\sqrt{\hbar}}{3\sqrt{2}\pi} k^{3/2} \eta^2. \tag{33}$$

In order to calculate the amplitude of the scalar fluctuations, we can use the variable $\mathcal{R}$ which measures the curvature perturbations and is related to v simply by [16, 17],

$$\mathcal{R}(\eta) = \frac{v}{z} \sim i \frac{\sqrt{2\pi^4 G^2 \hbar \rho_c}}{3} k^{-3/2}, \tag{34}$$

where we have only kept the dominant mode in terms of k, and also dropped the mode that decays as η grows. From this, it is possible to calculate the scalar power spectrum given by

$$\Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} |\mathcal{R}(\eta)|^2 \sim \frac{\pi^2 G^2 \hbar \rho_c}{9} k^0. \tag{35}$$

This shows that the scalar spectral index is $n_s = 1$, and thus the power spectrum is scale-invariant.

For the amplitude of the scalar perturbations given in (35) to agree with the observed value of $\Delta^2_{\mathcal{R}} \sim 10^{-9}$ [1], it is necessary for the critical density to be of the order of $10^{-9}\rho_{\rm Pl}$. This seems to contradict LQC where the critical energy density is usually assumed to be of the order of the Planck density, in which case the matter bounce scenario would be ruled out in LQC. However, it is important to remember that the numerical value of the critical energy density must ultimately be derived from loop quantum gravity, and this remains an open problem. The critical energy density may turn out to be much smaller than expected, in which case the matter bounce scenario would be viable in LQC.

C. The Equation of State $P = -\epsilon\rho$

A slight red tilt to the spectrum of scalar perturbations is obtained by working with the equation of state $P = -\epsilon\rho$, where $0 < \epsilon \ll 1$. We will not go through all of the details of the calculation here as they are a straightforward extension of what is done earlier in this section.

It is a relatively simple task to determine the spectrum of scalar perturbations for this new equation of state (the calculations simplify considerably if terms of order ϵ^2 and higher are dropped). The resulting spectrum has a scalar index of

$$n_s = 1 - 12\epsilon, \tag{36}$$

which given the observed scalar index of $n_s = 0.968 \pm 0.012$ [1], implies that we must have

$$\epsilon \approx 0.003, \tag{37}$$

which validates the approximation $\epsilon \ll 1$.

Thus, it is possible to match the observed red tilt of the scalar perturbation spectrum by choosing an appropriate matter field. In the next section, where we shall study tensor perturbations, we will exclusively work with a background where the equation of state is $P = -\epsilon\rho$, as this is the relevant setting whose predictions agree with observations.

IV. TENSOR PERTURBATIONS

For tensor perturbations, we will follow the same procedure as in the previous section, i.e., impose quantum vacuum initial conditions before the bounce and use matching conditions in order to determine the form of the tensor perturbations after the bounce.

One difference with the previous section is that we will work in the setting where the equation of state is $P = -\epsilon\rho$, with $0 < \epsilon \ll 1$. This is the background that gives a slight red tilt to the spectrum of scalar perturbations. In addition, as we shall see, for tensor perturbations the limit of $\epsilon = 0$ gives a qualitatively different result so it is important to work from the start with a nonzero value for ϵ. Because ϵ is very small, we will neglect all terms that are of the order of ϵ^2 or smaller.

To first order in ϵ, the scale factor and the proper time in the classical theory, in terms of the conformal time, are given by

$$a(\eta) = \left((1 - 6\epsilon) \frac{2\pi G \rho_c}{3} \eta^2 \right)^{1+3\epsilon}, \tag{38}$$

$$t(\eta) = (6\pi G \rho_c)^{1+3\epsilon} \left(\frac{1 - 2\epsilon}{3} \eta \right)^{3(1+2\epsilon)}. \tag{39}$$

While it is possible to expand the exponents around $\epsilon = 0$, this is not necessary and would merely complicate calculations later.

A. The Contracting Branch and the Bounce

In the classical limit, the Mukhanov-Sasaki equation for tensor perturbations with an equation of state $P = -\epsilon\rho$ is given by

$$\mu'' + \left(k^2 - \frac{2(1 + 9\epsilon)}{\eta^2} \right) \mu = 0, \tag{40}$$

and choosing the initial conditions to be the quantum vacuum state (21) gives

$$\mu(\eta) = \sqrt{\frac{-\pi\hbar\eta}{4}} H^{(1)}_n(-k\eta) \tag{41}$$

in the classical regime of the contracting branch of the cosmology with

$$n = \frac{3}{2} + 6\epsilon. \tag{42}$$

As before, we will link the solutions in the contracting and expanding branches via the leading order solution in k by using the integral form of the modified Mukhanov-Sasaki equation,

$$\mu(\eta) = D_1 z_T + D_2 z_T \int^{\eta} \frac{\mathrm{d}\bar{\eta}}{z_T^2} - k^2 z_T \int^{\eta} \frac{\mathrm{d}\bar{\eta}}{z_T^2} \int^{\bar{\eta}} \mathrm{d}\bar{\bar{\eta}}\, z_T\, \mu + \frac{2k^2}{\rho_c} z_T \int^{\eta} \frac{\mathrm{d}\bar{\eta}}{z_T^2} \int^{\bar{\eta}} \mathrm{d}\bar{\bar{\eta}}\, \rho\, z_T\, \mu, \tag{43}$$

and solving for the two leading order terms in k.

For $\omega = -\epsilon$ in LQC,

$$z_T(t) = \frac{\left[6\pi G\rho_c(1-2\epsilon)t^2 + 1\right]^{\frac{5}{2}+\frac{3\epsilon}{2}}}{\sqrt{6\pi G\rho_c(1-2\epsilon)t^2 - 1}}, \tag{44}$$

and therefore we find that the two leading order terms in k of μ are

$$\mu(t) = D_1 z_T(t) - \left[\frac{\epsilon}{\sqrt{6\pi G\rho_c}}\left(\arctan\sqrt{6\pi G\rho_c}\, t + \frac{\pi}{2}\right) + \frac{(1-\epsilon)t}{\left[6\pi G\rho_c(1-2\epsilon)t^2+1\right]^{1+\epsilon}}\right] D_2 z_T(t), \tag{45}$$

where the integration constant is again chosen in order to simplify the matching procedure. Then, in the regime $t \ll -1/\sqrt{6\pi G\rho_c}$, the expression for $\mu(t)$ simplifies to

$$\mu(t) = D_1 a(t) - \frac{(1-2\epsilon)D_2 a(t)}{[6\pi G\rho_c(1-2\epsilon)]^{1+\epsilon} t^{1+2\epsilon}}, \tag{46}$$

and therefore, for the two solutions to match, we must have

$$D_1 = \sqrt{\frac{8\hbar}{9}} \left(\frac{3}{8\pi G\rho_c}\right)^{1+3\epsilon} k^n, \tag{47}$$

$$D_2 = i\sqrt{\frac{9\hbar}{32}} \left(\frac{8\pi G\rho_c}{3}\right)^{1+3\epsilon} k^{-n}, \tag{48}$$

where we have dropped the nonexponential dependence of ϵ in the prefactors as this will not be relevant for our calculations.

B. The Expanding Branch

In order to determine the spectrum of gravitational waves in the classical regime of the expanding branch after the bounce, it is necessary to match the classical solution

$$\mu(\eta) = \sqrt{\eta}\left[E_1 H_n^{(1)}(k\eta) + E_2 H_n^{(2)}(k\eta)\right] \tag{49}$$

to leading order in k with the leading order terms of the formal solution (45) in the regime $t \gg 1/\sqrt{6\pi G\rho_c}$, where the two leading order terms have the form

$$\mu(t) = \left(D_1 - \sqrt{\frac{\pi}{6G\rho_c}}\,\epsilon D_2\right) a(t) - \frac{(1-2\epsilon)D_2 a(t)}{[6\pi G\rho_c(1-2\epsilon)]^{1+\epsilon} t^{1+2\epsilon}}. \tag{50}$$

From this expression, and the previous calculations, one immediately sees that

$$\mu(\eta) = \sqrt{\pi\hbar\eta}\left[-i\frac{\epsilon}{32\, k^{2n}}\sqrt{\frac{27\pi}{2G\rho_c}}\left(\frac{8\pi G\rho_c}{3}\right)^{2+6\epsilon} J_n(k\eta) + \frac{1}{2}H_n^{(2)}(k\eta)\right], \tag{51}$$

where the prefactors are understood to be accurate to leading order in k. Again, just as in the case of scalar perturbations, a new mode is created by the bounce. As the amplitude of the tensor perturbations is of the order of ϵ, this will give a small tensor to scalar ratio.

From the expression of μ, it is possible to determine h,

$$h = \frac{\mu}{a} \sim -i\frac{\epsilon\pi}{2}\sqrt{\frac{\hbar}{2}}\left(\frac{8\pi G\rho_c}{3}\right)^{\frac{1}{2}+3\epsilon} k^{-\frac{3}{2}-6\epsilon}, \tag{52}$$

where the first equality holds in the classical regime and we have kept only the constant part of the leading order term in k on the righthand side.

The amplitude of the tensor perturbations is then given by[2]

$$\Delta_h^2(k) = 64\pi G \frac{k^3}{2\pi^2}|h|^2 = \epsilon^2\, \frac{32\pi^2 G^2\hbar\rho_c}{3}\left(\frac{8\pi G\rho_c}{3k^2}\right)^{6\epsilon}, \tag{53}$$

and thus we get a spectrum of tensor perturbations with a small amplitude and an almost scale-invariant spectrum with $n_T = -12\epsilon$.

Finally, it is possible to determine the tensor to scalar ratio, given by

$$r = \frac{\Delta_h^2}{\Delta_{\mathcal{R}}^2} = 96\,\epsilon^2 \approx 9\times 10^{-4}, \tag{54}$$

where we have inserted the value of ϵ in (37), which was determined by the observed tilt of the spectrum of scalar perturbations. The predicted value for r is so small that, if the matter bounce scenario is the correct one, we do not expect to observe a primordial gravitational wave background until the precision of astronomical observations increases by an order of magnitude or two.

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[2] There is an extra factor of $32\pi G$ in the definition of Δ_h^2 for dimensional reasons, and another additional factor of 2 to account for the two polarizations. See e.g. [2] for details.

It is worth noting that it is important to work with the equation of state $P = -\epsilon\rho$ for tensor perturbations as the predictions are significantly different than for a pure dust matter field with $P = 0$: for the case $P = 0$, no new mode appears in (51) and then the resulting spectrum would have a strong blue tilt of $n_T = 6$ (and therefore primordial gravitational waves would not be observable as their amplitude would be proportional to $\ell_{\rm Pl}^6 k_0^6$, an extremely small factor for all relevant modes). Thus, the small deviation from a dust fluid plays an important role as it generates the new mode in (51) which is scale-invariant and the only one which is potentially observable today in this scenario.

V. DISCUSSION

In loop quantum cosmology, quantum gravity effects modify both the Friedmann equations and the Mukhanov-Sasaki equations, which govern the dynamics of the homogeneous background and of the linear perturbations respectively. These modifications become important at large curvature scales, and especially at the bounce point. Therefore, it is important to include these effects in any setting where the cosmological bounce plays a role, and this is what has been done here for the matter bounce scenario.

The matter bounce in loop quantum cosmology gives a scale-invariant spectrum for scalar and tensor perturbations, and a slight red tilt is obtained for the equation of state $P = -\epsilon\rho$, with $0 < \epsilon \ll 1$. The same tilt is predicted for the scalar and tensor modes. As an identical tilt for the scalar and tensor modes is not expected in inflationary models, this is one way the two scenarios can be differentiated.

The other main results are that the observed amplitude for scalar perturbations is obtained for a value of the critical energy density of $\rho_c \sim 10^{-9}\rho_{\rm Pl}$, and a tensor to scalar ratio is predicted to be $r \sim 9 \times 10^{-4}$. Note that there are some important differences between these predictions for the matter bounce scenario and those given in [5] where quantum gravity effects were not included, particularly regarding the amplitudes of the spectra and their relative importance.

Although the value of ρ_c is usually assumed to be within one order of magnitude of the Planck energy density, this is a quantity that should be derived from loop quantum gravity. Thus, until the relation between the full theory of LQG and its cosmological sector is better understood, the critical energy density remains an unknown, and may turn out to be considerably smaller than expected. In any case, it is clear that for the matter bounce scenario to be viable in LQC we must have $\rho_c \sim 10^{-9}\rho_{\rm Pl}$.

One of the key predictions here is the small tensor to scalar ratio, which is proportional to ϵ^2 (and is therefore related to n_s). The modifications to the scalar and tensor Mukhanov-Sasaki equations are slightly different, and this plays a major role in the matter bounce scenario as one of its effects is the much smaller amplitude of the tensor perturbations than that predicted from classical considerations. The reason why the amplitude of the tensor perturbations is so small (and vanishes in the limit $\epsilon \to 0$) is that z_T, defined in (15), becomes imaginary near the bounce. Therefore, the integral $\int {\rm d}\eta/z_T^2$ can vanish as some portions of it will be negative. This is one way that quantum gravity effects can significantly change results obtained in a purely classical setting.

It is reasonable to expect that important modifications may also arise in other cosmological models, including inflation. As it is known how to include inflation in LQC [18, 19], it is important now to study how quantum gravity effects could affect the standard inflationary predictions. Some work has already been done in this direction, for example [15, 20].

As the effects of LQC and other quantum cosmology models on the CMB and primordial gravitational waves are better understood, and the observations continue to improve, it will become possible to differentiate between (i) alternative cosmological scenarios, and (ii) the various quantum cosmology theories, and thus determine which combination of the two is realized in nature.

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